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Anti fuzzy ideals in BE-algebra

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Abstract. In this paper, we apply the Biswas idea to BE-algebras and introduce the notion of an anti fuzzy ideal in BE-algebras. Furthermore, these sets are considered in the context of transitive and self distributive BE-algebras and their ideals, providing characterizations of one type, the generalized lower sets, in other type, ideals.

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1. Introduction

The concept of fuzzy sets was first initiated by Zadeh [11] 1965. Since then these ideas have been applied to other algebraic structures such as semigroup, group, ring, etc. Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [6, 7]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Jun et al., [8] introduced the notion of BH-algebra, which is a generalization of BCH/BCI/BCK-algebras. In [10], Kim and Kim introduced the notion of a BE-algebra as a dualization of generalization of a BCK-algebra. In 1990, S. Biswas introduced the concept of anti fuzzy subgroup of group [4]. Recently, Hong and Jun, modifying Biswas idea, apply the concept to BCK-algebras. So, they defined the notion of anti fuzzy ideal of BCK algebras and obtain some useful results on it. In [9] Jun and Song introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. Further more see [1, 2].

In this paper, we apply the Biswas idea to BE-algebras, and introduce the concept of anti fuzzy ideal in BE-algebras and investigate some related properties. Also we characterize anti fuzzy ideals in BE-algebras.

2. Preliminaries

We recall some definition and results [3, 9, 10].

Definition 2.1. An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE-algebra [10] if

$$x * x = 1 \text{ for all } x \in X, \tag{2.1}$$

$$x * 1 = 1 \text{ for all } x \in X, \tag{2.2}$$

$$1 * x = x \text{ for all } x \in X, \tag{2.3}$$

$$x * (y * z) = y * (x * z) \text{ for all } x, y, z \in X. \tag{2.4}$$

A relation "$\leq$" on a BE-algebra X is defined by

$$(\forall x, y \in X)\ (x \leq y \iff x * y = 1). \tag{2.5}$$

A BE-algebra $(X; *, 1)$ is said to be transitive [3] if it satisfies:

$$(\forall x, y, z \in X)\ (y * z \leq (x * y) * (x * z)). \tag{2.6}$$

A BE-algebra $(X; *, 1)$ is said to be self distributive [10] if it satisfies:

$$(\forall x, y, z \in X)(x * (y * z) = (x * y) * (x * z)). \tag{2.7}$$

Note that every self distributive BE-algebra is transitive, but the converse is not true in general [3]

Example 2.2 ([10])**.** Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 2.3 ([10])**.** A BE-algebra $(X; *, 1)$ is said to be self distributive if

$$x * (y * z) = (x * y) * (x * z) \textit{ for all } x, y, z \in X.$$

Example 2.4 ([10]). Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE-algebra satisfying self distributivity. Note that the BE-algebra in Example 2.2 is not self distributive, since $d*(a*0) = d*d = 1$, while $(d*a)*(d*0) = 1*a = a$.

Definition 2.5 ([9]). A non-empty subset I of X is called an ideal of X if

$$\forall x \in X \text{ and } \forall a \in I \implies x * a \in I, \text{ i.e., } X * I \subseteq I, \tag{2.8}$$

$$\forall x \in X, \forall a, b \in I \text{ imply } (a*(b*x))*x \in I. \tag{2.9}$$

In Example 2.2, $\{1, a, b\}$ is an ideal of X, but $\{1, a\}$ is not an ideal of X, since $(a*(a*b))*b = (a*a)*b = 1*b = b \notin \{1; a\}$.

It was proved that every ideal I of a BE-algebra X contains 1, and if $a \in I$ and $x \in X$, then $(a*x)*x \in I$. Moreover, if I is an ideal of X and if $a \in I$ and $a \leq x$, then $x \in I$ [9].

3. Major section

In this section we introduce anti fuzzy ideals in BE-algebras and discuss some fundamental results.

Definition 3.1. A fuzzy subset f of a BE-algebra X is called an anti fuzzy ideal of X if it satisfies;

$$(\forall x, y \in X) f(xy) \leq f(y) \tag{3.1}$$

$$(\forall x, y, z \in X)(f((x*(y*z))*z) \leq \max\{f(x), f(y)\}). \tag{3.2}$$

Example 3.2. Consider the BE-algebra X described in Example 2.2. Now we define a fuzzy set f on X as:

$$f(x) = \begin{cases} 0.4 \text{ if } x \in \{1, a, b\} \\ 0.7 \text{ if } x \in \{c, d, 0\} \end{cases}$$

Then, by routine calculation f is an anti fuzzy ideal of X. Now we define a fuzzy set on X as :

$$f(x) = \begin{cases} 0.4 \text{ if } x \in \{1, a\} \\ 0.7 \text{ if } x \in \{b, c, d, 0\} \end{cases}$$

Then, f is a not an anti fuzzy ideal of X, i.e.,

$$f((a*(a*b))*b) = f(b) = 0.7 > 0.4 = \max\{f(a), f(a)\}.$$

Theorem 3.3. *Let f be a fuzzy set in X. Then f is an anti fuzzy ideal of X if and only if it satisfies:*

$$(\forall \alpha \in [0,1])(L(f;\alpha) \neq \varnothing \implies L(f;\alpha) \text{ is an ideal of } X), \tag{3.3}$$

where $L(f;\alpha) := \{x \in X \mid f(x) \leq \alpha\}$.

Proof. Let f be an anti fuzzy ideal in X. Let $\alpha \in [0,1]$ be such that $L(f;\alpha) \neq \varnothing$. Let x,$y \in X$ be such that $y \in L(f;\alpha)$. Then $f(y) \leq \alpha$, and so $f(x * y) \leq f(y) \leq \alpha$. Thus $x * y \in L(f;\alpha)$. Let $x \in X$ and $a, b \in L(f;\alpha)$. Then $f(a) \leq \alpha$, $f(b) \leq \alpha$ and we have

$$(f((a*(b*x))*x) \leq \max\{f(a), f(b)\}) \leq \alpha$$

so that $(a*(b*x))*x) \in L(f;\alpha)$. Hence $L(f;\alpha)$ is an ideal of X.

Conversely, suppose that f satisfies (3). If $f(a*b) > f(b)$ for some $a, b \in X$, then $f(a*b) > \alpha_0 > f(b)$ by taking $\alpha_0 := (f(a*b) + f(b))/2$. Hence $a*b \notin U(f;\alpha_0)$ and $b \in U(f;\alpha_0)$, which is a contradiction. Let $a, b, c \in X$ be such that

$$f((a*(b*x))*x) > \max\{f(a), f(b)\}.$$

Taking $\beta_0 = (f((a*(b*x))*x) + \max\{f(a), f(b)\})$, we have $\beta_0 \in [0,1]$ and

$$f((a*(b*x))*x) > \beta_0 > \max\{f(a), f(b)\}.$$

it follows that $a, b \in U(f;\beta_0)$ and $(a*(b*x)) \notin U(f;\beta_0)$. This is a contradiction and therefore f is an anti fuzzy ideal of X. □

Lemma 3.4. *Every anti fuzzy ideal of X satisfies the following inequality:*

$$(\forall x \in X)(\mu(1) \leq \mu(x)). \tag{3.4}$$

Proof. Since in BE-algebra we have $x * x = 1$, thus we have

$$\mu(1) = \mu(x*x) \leq \mu(x)$$

for all $x \in X$. □

Proposition 3.5. *If f is an anti fuzzy ideal of X, then*

$$(\forall x, y \in X)(f((x*y)*y) \leq f(x)).$$

Proof. Taking $y = 1$ and $z = y$ in (2), we get

$$f((x*y)*y) = f((x*(1*y))*y) \leq \max\{f(x), f(1)\} = f(x)$$

for all $x, y \in X$. □

Corollary 3.6. *Every anti fuzzy ideal f of X is reverse order preserving, that is, f satisfies:*

$$(\forall x, y \in X)(x \leq y \implies f(x) \geq f(y)). \tag{3.5}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f(y) = f(1*y) = f((x*y)*y) \leq f(x)$$

by (2.3) and (3.5). □

Proposition 3.7. *Let f be a fuzzy set in X which satisfies* (3.4) *and*

$$(\forall x, y, z \in X)(f(x * z) \leq \max\{f(x * (y * z)), f(y)\}). \tag{3.6}$$

Then, f is reverse order preserving.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f(y) = f(1 * y) \leq \max\{f(1 * (x * y)), f(x)\} = \max\{f(1 * 1)), f(x)\}$$

by (2.1), (2.3), (3.7) and (3.4). □

Theorem 3.8. *Let X be a transitive BE-algebra. A fuzzy set f in X is an anti fuzzy ideal of X if and only if it satisfies conditions* (3.4) *and* (3.7).

Proof. Let f be an anti fuzzy ideal of X. By lemma (3.1), f satisfies (3.4). Since X is transitive, we have

$$(y * z) * z \leq (x * (y * z)) * (x * z), \tag{3.7}$$

i.e., $((y * z) * z)((x * (y * z)) * (x * z)) = 1$ for all $x, y, z \in X$. It follows from (2.3), (3.2) and Proposition 3.1 that

$$\begin{aligned} f(x * z) &= f(1 * (x * z)) \\ &= f(((y * z) * z)((x * (y * z)) * (x * z)) * (x * z)) \\ &\leq \max\{f((y * z) * z), f(x * (y * z))\} \\ &\leq \max\{f(x * (y * z)), f(y)\}. \end{aligned}$$

Hence, f satisfies (3.7). Conversely suppose that f satisfies two conditions (3.4) and (3.7). Using (3.7), (2.1), (2.2) and (3.4), we have

$$\begin{aligned} f(x * y) &\leq \max\{f(x * (y * y)), f(y)\} \\ &= \max\{f(x * 1), f(y)\} \\ &= \max\{f(1), f(y)\} \\ &= f(y) \end{aligned}$$

and

$$\begin{aligned} f((x * y) * y) &\leq \max\{f((x * y) * (x * y)), f(x)\} \\ &= \max\{f(1), f(x)\} = f(x) \end{aligned} \tag{3.8}$$

for all $x, y \in X$. Since f is reverse order preserving by Proposition 3.2, it follows from (3.8) that

$$f((y * z) * z) \geq f((x * (y * z)) * (x * z))$$

and so from (3.7) and (3.10) that

$$\begin{aligned} f((x * (y * z)) * z) &\leq \max\{f(((x * (y * z)) * (x * z)), f(x)\} \\ &\leq \max\{f((y * z) * z), f(x)\} \\ &\leq \max\{f(x), f(y)\} \end{aligned}$$

for all $x, y, z \in X$. Hence, f is a fuzzy ideal of X. □

Corollary 3.9. *Let X be a self distributive BE-algebra. A fuzzy set f in X is an anti fuzzy ideal if and only if it satisfies condition* (3.4) *and* (3.7).

Proof. Straightforward. □

For every $a, b \in X$, let f_a^b be a fuzzy set in X defined by

$$f_a^b := \begin{cases} \alpha \text{ if } a * (b * c) = 1 \\ \beta \text{ otherwise} \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [0, 1]$ with $\alpha < \beta$.

The following example shows that there exist $a, b \in X$ such that f_a^b is not an anti fuzzy ideal of X.

Example 3.10. Let $X = \{1, a, b, c\}$ with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then, $(X; *, 1)$ is a BE-algebra [3]. But f_b^1 is not an anti fuzzy ideal of X since

$$\begin{aligned} f_b^1((a * (a * c)) * c) &= f_b^1((a * a) * c) = f_b^1(1 * c) \\ &= f_b^1(c) = \beta > \alpha \\ &= \max\{f_b^1(a), f_b^1(a)\}. \end{aligned}$$

Theorem 3.11. *If X is self distributive, then the fuzzy set f_a^b in X is an anti fuzzy ideal of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. For every $x, y \in X$, if $a*(b*y) \neq 1$, then $f_a^b(y) = \beta \geq f_a^b(x*y)$. Assume that $a * (b * y) = 1$. Then

$$\begin{aligned} a * (b * (x * y)) &= a * ((b * x) * (b * y)) \\ &= (a * (b * x)) * (a * (b * y)) \\ &= (a * (b * x)) * 1 = 1, \end{aligned}$$

and so $f_a^b(x * y) = \alpha = f_a^b(y)$. Hence $f_a^b(x * y) \leq f_a^b(y)$ for all $x, y \in X$. Now, for every $x, y, z \in X$, if $a * (b * x) \neq 1$ or $a * (b * y) \neq 1$, then $f_a^b(x) = \beta$ or $f_a^b(y) = \beta$ Thus

$$f_a^b((x * (y * z)) * z) \leq \beta = \max\{f_a^b(x), f_a^b(y)\}.$$

Suppose that $a * (b * x) = 1$ and $a * (b * y) = 1$. Then

$$\begin{aligned} a * (b * ((x * (y * z)) * z)) &= a * ((b * ((x * (y * z))) * (b * z)) \\ &= a * (b * ((x * (y * z)))) * (a * (b * z)) \\ &= ((a * (b * x)) * (a * (b * (y * z)))) * (a * (b * z)) \\ &= (1 * (a * (b * (y * z)))) * (a * (b * z)) \\ &= (a * (b * (y * z)))) * (a * (b * z)) \\ &= ((a * (b * y)) * (a * (b * z))) * (a * (b * z)) \\ &= (1 * (a * (b * z))) * (a * (b * z)) \\ &= (a * (b * z)) * (a * (b * z)) = 1 \end{aligned}$$

which implies that $f_a^b((x * (y * z)) * z) = \alpha < \beta = \max\{f_a^b(x), f_a^b(y)\}$.

Therefore, $f_a^b((x*(y*z))*z) \leq \max\{f_a^b(x), f_a^b(y)\}$ for all $x, y, z \in X$. Consequently, f_a^b is an anti fuzzy ideal of X for all $a, b \in X$. $\square$

For any $a, b \in X$, the set $A(a, b) := \{x \in X \mid a*(b*x) = 1\}$ is called the upper set of a and b [3]. Clearly , $1, a, b \in A(a, b)$ for all $a, b \in X$ [3].

Lemma 3.12 ([9]). *A nonempty subset I of X is an ideal of X if and only if it satisfies*

$$1 \in I$$
$$(\forall x, z \in X)(\forall y \in X)(x*(y*z) \in I \implies x*z \in I).$$

Theorem 3.13. *Let f be a fuzzy set in X. Then f is an anti fuzzy ideal of X if and only if it satisfies:*

$$(\forall a, b \in X)(\forall \alpha \in [0,1])(a, b \in L(f;\alpha)).$$

Proof. Suppose that f is an anti fuzzy ideal of X and let $a, b \in L(f;\alpha)$. Then $f(a) \leq \alpha$ and $f(b) \leq \alpha$. Let $x \in A(a, b)$. Then, $a*(b*x) = 1$. Hence,

$$f(x) = f(1*x) = f((a*(b*x))*x) \leq \max\{f(a), f(b)\} \leq \alpha,$$

and so $x \in L(f;\alpha)$. Thus $A(a, b) \subseteq L(f;\alpha)$.

Conversely, since $1 \in A(a, b) \subseteq L(f;\alpha)$ thus for all $a, b \in X$. Let $x, y, z \in X$ be such that $x*(y*z) \in L(f;\alpha)$ and $y \in L(f;\alpha)$. Since

$$(x*(y*z))*(y*(x*z)) = (x*(y*z))(x*(y*z)) = 1$$

by 2.4 and 2.1, we have $x*z \in A(x*(y*z), y) \subseteq L(f;\alpha)$. It follows from Lemma 3.2 that $L(f;\alpha)$ is an anti fuzzy ideal of X. Hence f is an anti fuzzy ideal of X by Theorem 3.1. $\square$

Corollary 3.14. *If f is an anti fuzzy ideal of X, then*

$$(\forall \alpha \in [0,1])\ (L(f;\alpha) \neq \varnothing \implies L(f;\alpha) = \bigcup_{a,b \in L(f;\alpha)} A(a,b)).$$

Proof. Let $\alpha \in [0,1]$ be such that $L(f;\alpha) \neq \varnothing$. Since, we have

$$L(f;\alpha) \subseteq \bigcup_{a \in L(f;\alpha)} A(a,1) \subseteq \bigcup_{a,b \in L(f;\alpha)} A(a,b).$$

Now let $x \in \bigcup_{a,b \in L(f;\alpha)} A(a,b)$. Then, there exist $u, v \in L(f;\alpha)$ such that $x \in A(u,v) \subseteq L(f;\alpha)$ by Theorem 4. Thus $\bigcup_{a,b \in L(f;\alpha)} A(a,b) \subseteq L(f;\alpha)$. This completes the proof. $\square$

4. Conclusions

Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [6, 7]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebra. Kim and Kim defined a new class of algebra: called BE-algebra in [10]. In this article we studied ideal theory of BE-algebra in context of fuzzy set to introduced anti fuzzy ideals in BE-algebras. We discussed some characterizations of BE-algebras in terms of anti-fuzzy ideals. We also discussed some basic properties of BE-algebras in terms of these notions which are necessary for further study of BE-algebras. We will be focus on further study in BE-algebras in terms of fuzzy sets as follows: We will defined further generalization of anti fuzzy ideals in BE-algebra. We will study BE-algebra in terms of rough set theory. We will define rough fuzzy ideals in BE-algebras.

References

[1] S. Abdullah, M. Aslam, M. Amin and T. Khan, Direct product of finite fuzzy subsets of LA-semigroups, Ann. Fuzzy Math. Inform. 3(2) (2012) 281–292.
[2] S. Abdullah, M. Aslam, T. A. Khan and M. Naeem, A new type of fuzzy normal subgroups and fuzzy cosets, J. Intell. Fuzzy Syst. 25 (2013) 37–47.
[3] S. S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, Sci. Math. Jpn. 68 (2008) 279–285.
[4] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets and Systems 35 (1990) 121–124.
[5] Q. P. Hu and X. Li, On BCH-algebras, Math. Sem. Notes, Kobe Univ. 11 (1983) 313–320.
[6] Y. Imai, On BCI-algebras, Math. Sem. Notes, Kobe Univ. 8 (1980) 125–130.
[7] K. Iseki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 23(1) (1978/79) 1–26.
[8] Y. B. Jun, E. H. Roh, and H. S. Kim, On BH-algebras, Sci. Math. 1 (1998) 347–351.
[9] Y. B. Jun, K. J. Lee and S. Z. Song. Fuzzy ideals in BE-algebra, Bull. Malays. Math. Sci. Soc. 33 (2010) 147–153.
[10] H. S. Kim and Y. H. Kim, On BE-algebras, Sci. Math. Jpn. 66 (2007) 113–116.
[11] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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